

AP PRECALCULUS STUDY GUIDE

180 High-Yield Concepts

Advanced Placement Course | College Board CED-aligned

- All 4 CED units: Polynomials to Vectors & Matrices
- Every concept: explanation + high-yield exam tip + example
- Polynomial, exponential, trig, polar, parametric & matrix topics
- Perfect for the AP Precalculus exam (first given in 2024)

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How to Use This Guide

This guide condenses the AP Precalculus course and exam description into high-yield concepts covering all four College Board units. It is not a textbook replacement -- it is the fastest way to review everything the exam covers and to target your weak units.

The AP Precalculus exam is two sections: Part A (multiple choice) and Part B (free response), over about 3 hours, with a calculator required for a portion. The four units span polynomial and rational functions, exponential and logarithmic functions, trigonometric and polar functions, and functions involving parameters, vectors, and matrices.

How to use the concepts

- Each concept = term + concise explanation + high-yield exam tip (+ example where useful).
- First pass: read every concept in order. Underline anything unfamiliar.
- Second pass: focus only on the concepts you underlined.
- Two weeks before the exam: rapid-review every tip line (they are the highest-yield).
- Use the Table of Contents to jump straight to your weakest units.

Table of Contents

1	Polynomial and Rational Functions	57 concepts
2	Exponential and Logarithmic Functions	43 concepts
3	Trigonometric and Polar Functions	39 concepts
4	Functions Involving Parameters, Vectors, and Matrices	41 concepts
		Total: 180 high-yield concepts

How to use the TOC: work through the units in order on your first pass, then jump back to your weakest units for review. Each concept includes an Explanation, a high-yield exam tip, and where useful an Example. Mark the concepts you miss with a small dot in the margin and revisit them 48 hours later.

Unit 1: Polynomial and Rational Functions

30-40% of the exam. Rates of change, polynomial behavior, rational functions, and modeling.

1 Function Definition & Notation

Explanation: A function maps each input (domain) to exactly one output (range). $f(x)$ notation names the output for input x . The domain is the set of all valid inputs.

AP Precalc Tip: AP Precalc expects you to evaluate $f(a)$ and find the domain of a function. A function is only a function if each x has ONE output.

Example: $f(x) = x^2 + 1$ gives $f(3) = 10$. Domain of $f(x) = \sqrt{x}$ is $x \geq 0$.

2 Domain & Range

Explanation: The domain is all possible inputs; the range is all possible outputs. Exclude values that make a denominator zero or a square root of a negative.

AP Precalc Tip: To find a domain, set the denominator not equal to zero and the radicand ≥ 0 . Range answers usually follow from the graph or reasoning.

Example: $f(x) = 1/(x-2)$: domain is all real x except $x = 2$. $g(x) = \sqrt{x-3}$: domain is $x \geq 3$.

3 Function Representing Options

Explanation: Functions can be given as a formula, a table, a graph, or a verbal description. On the AP exam you must shift between these representations fluently.

AP Precalc Tip: If a table repeats an x with two different y values, it is NOT a function. Read the graph's labels carefully --- the axes matter.

Example: A table with $x = 2$ mapped to both 5 and 7 fails the vertical-line test, so it is not a function.

4 Average Rate of Change

Explanation: The average rate of change of f on $[a, b]$ is $(f(b) - f(a)) / (b - a)$. It equals the slope of the secant line through $(a, f(a))$ and $(b, f(b))$.

AP Precalc Tip: Average rate of change is the slope formula over an interval. It is a single number describing change on average, not at a point.

Example: For $f(x) = x^2$ on $[1, 3]$: $(9 - 1)/(3 - 1) = 8/2 = 4$.

5 Change in the Input & Output

Explanation: Change in input is Δx ; change in output is $\Delta y = f(x + \Delta x) - f(x)$. The ratio $\Delta y / \Delta x$ is the average rate of change over that Δx .

AP Precalc Tip: The AP exam frames much of precalc through rates of change: $\Delta y / \Delta x$ and how it behaves as Δx gets small.

Example: If x goes from 2 to 5 ($\Delta x = 3$) and f goes from 4 to 13, $\Delta y = 9$ and the average rate $= 3$.

6 Linear Functions

Explanation: A linear function has the form $f(x) = mx + b$ with a constant average rate of change m . The graph is a straight line. m is the slope, b the y -intercept.

AP Precalc Tip: Linear change means equal steps in x produce equal steps in y . If a table has constant first differences, the function is linear.

Example: $f(x) = 3x - 5$ has slope 3: every increase of 1 in x raises y by 3.

7 Rate of Change & Linearity

Explanation: A function is linear exactly when its average rate of change is constant over every interval. This is the defining property that the AP exam tests.

AP Precalc Tip: To test linearity from a table, check that the first differences (changes in y for equal x -steps) are constant.

Example: $x: 0,1,2,3 \mid y: 2,5,8,11$ --- first differences are all $+3$, so it is linear.

8 Graphing Linear Functions

Explanation: Plot the y-intercept, then use the slope as rise over run to find a second point. Draw the line through both. Slope can be positive, negative, zero, or undefined.

AP Precalc Tip: A zero slope is horizontal ($y = b$); an undefined slope is vertical ($x = c$) --- not a function of x . Read slope from a graph as rise/run.

Example: $f(x) = -2x + 4$ passes through (0,4) and (1,2); the slope -2 means it falls quickly.

9 Parallel & Perpendicular Lines

Explanation: Parallel lines have equal slopes. Perpendicular lines have slopes that are negative reciprocals: $m_1 * m_2 = -1$. A vertical line is perpendicular to every horizontal line.

AP Precalc Tip: Perpendicular means the slopes multiply to -1. A horizontal line (slope 0) and vertical line (undefined) are perpendicular.

Example: Lines with slopes 2 and $-1/2$ are perpendicular because $2 * (-1/2) = -1$.

10 Forms of a Line

Explanation: Point-slope: $y - y_1 = m(x - x_1)$. Slope-intercept: $y = mx + b$. Standard: $Ax + By = C$. Convert between them as needed.

AP Precalc Tip: Point-slope is the fastest when you know one point and the slope. Be ready to write the equation of a line through two points.

Example: Through (2,3) with $m = 4$: $y - 3 = 4(x - 2)$. Rearranged: $y = 4x - 5$.

11 Horizontal & Vertical Lines

Explanation: Horizontal: $y = k$ (constant, slope 0). Vertical: $x = k$ (not a function, undefined slope). Vertical lines fail the vertical line test.

AP Precalc Tip: A vertical line $x = k$ is NOT a function of x --- the same x maps to infinitely many y values. This is a frequent trap.

Example: $y = 7$ is horizontal; $x = 7$ is vertical.

12 Piecewise Functions

Explanation: A piecewise function uses different rules on different intervals of the domain. Evaluate using the rule that matches the input's interval.

AP Precalc Tip: Check the interval boundaries carefully --- which rule applies at the breakpoint ($\leq$ vs $<$) changes the answer. Verify continuity at the joint.

Example: $f(x) = \{ x \text{ if } x < 2; 3 \text{ if } x \geq 2 \}$. Then $f(1) = 1$ and $f(2) = 3$.

13 Absolute Value Function

Explanation: $f(x) = |x|$ returns the distance from x to 0. The graph is a V shape with vertex at the origin, slope -1 for $x < 0$ and +1 for $x > 0$.

AP Precalc Tip: Absolute value makes all outputs nonnegative. The V shape has a sharp corner (no derivative there) --- a classic AP graph.

Example: $|x| = |x - 0|$. Solve $|x - 3| = 5$: $x = 8$ or $x = -2$ (distance 5 from 3).

14 Greatest Integer (Floor) Function

Explanation: $f(x) = \text{floor}(x)$ returns the greatest integer $\leq x$. The graph is a step function with jumps at integer values of x .

AP Precalc Tip: $\text{floor}(x)$ rounds DOWN, even for negatives: $\text{floor}(-1.7) = -2$. For $2 \leq x < 3$, $\text{floor}(x) = 2$.

Example: $\text{floor}(2.9) = 2$, $\text{floor}(3) = 3$, $\text{floor}(-0.4) = -1$.

15 Even & Odd Functions

Explanation: Even: $f(-x) = f(x)$, symmetric about the y-axis (e.g., x^2). Odd: $f(-x) = -f(x)$, symmetric about the origin (e.g., x^3). Many functions are neither.

AP Precalc Tip: Test even/odd by plugging in $-x$ and simplifying. Even + powers are symmetric; knowing symmetry speeds up graphing and limits.

Example: $f(x) = x^4 - x^2$ is even; $g(x) = x^3$ is odd; $h(x) = x^2 + x$ is neither.

16 Increasing & Decreasing Intervals

Explanation: A function is increasing where larger x gives larger y , and decreasing where larger x gives smaller y . State these as x -intervals.

AP Precalc Tip: Use interval notation for where the function rises or falls. Be careful --- endpoints of intervals are convention; open vs closed matters little on AP.

Example: $f(x) = x^2$ is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$.

17 Function Symmetry & Intercepts

Explanation: Find x-intercepts by solving $f(x) = 0$; find the y-intercept by evaluating $f(0)$. Use symmetry to find multiple intercepts faster.

AP Precalc Tip: The y-intercept is $f(0)$, always $(0, f(0))$. x-intercepts may be many --- factor and solve. On graphs, read them where the curve crosses the axes.

Example: $f(x) = x^2 - 4$ has x-intercepts at $x = -2$ and $x = 2$, and y-intercept at $(0, -4)$.

18 End Behavior of Polynomials

Explanation: As $|x|$ grows large, a polynomial behaves like its leading term $a \cdot x^n$. If n is even, both ends go the same direction; if n is odd, opposite ends. Sign of a sets direction.

AP Precalc Tip: End behavior depends ONLY on the leading term's degree and sign. Even degree + positive a = up on both ends; odd + positive a = down/up.

Example: $f(x) = -2x^4 + \dots$ falls on both ends (even degree, negative leading coefficient). $g(x) = 5x^3$ rises on the right, falls on the left.

19 Quadratic Functions & Vertex Form

Explanation: $f(x) = a(x - h)^2 + k$ has vertex (h, k) and axis of symmetry $x = h$. If $a > 0$ it opens up; if $a < 0$ it opens down.

AP Precalc Tip: Complete the square or use $h = -b/(2a)$ to find the vertex. The vertex is the max ($a < 0$) or min ($a > 0$) of the quadratic.

Example: $f(x) = 2(x - 1)^2 + 3$ has vertex $(1, 3)$ and opens up, so 3 is the minimum value.

20 Roots / Zeros of Quadratics

Explanation: Solve $ax^2 + bx + c = 0$ by factoring, completing the square, or the quadratic formula $x = [-b \pm \sqrt{b^2 - 4ac}] / (2a)$.

AP Precalc Tip: The discriminant $b^2 - 4ac$ tells the number of real roots: positive = two, zero = one, negative = zero real roots.

Example: $x^2 - 5x + 6 = 0$ factors to $(x-2)(x-3) = 0$, giving $x = 2$ and $x = 3$.

21 Discriminant & Nature of Roots

Explanation: The discriminant $D = b^2 - 4ac$ determines real vs complex roots. $D > 0$: two distinct real roots; $D = 0$: one repeated real root; $D < 0$: two complex roots.

AP Precalc Tip: A negative discriminant means the graph never crosses the x-axis. The number of x-intercepts equals the number of distinct real roots.

Example: $x^2 + 1 = 0$ has $D = -4 < 0$, so it has only complex roots (no x-intercepts).

22 Sum & Product of Roots

Explanation: For $ax^2 + bx + c$, the sum of roots is $-b/a$ and the product is c/a . Useful for quick checks and for building quadratics from roots.

AP Precalc Tip: Use $\text{sum} = -b/a$ and $\text{product} = c/a$ to reconstruct a quadratic from its roots: $x^2 - (\text{sum})x + (\text{product}) = 0$.

Example: Roots 3 and -1 give sum 2 and product -3, so the quadratic is $x^2 - 2x - 3$.

23 Higher-Degree Polynomials

Explanation: Degree n polynomials have at most n real roots and at most $n-1$ turning points. Their graphs are smooth and continuous curves.

AP Precalc Tip: The number of turning points is at most degree - 1. A cubic (degree 3) has at most 2 turning points and up to 3 x-intercepts.

Example: A fourth-degree polynomial can cross the x-axis up to 4 times and has up to 3 turning points.

24 Factor Theorem & Remainder Theorem

Explanation: If $f(c) = 0$, then $(x - c)$ is a factor (Factor Theorem). The Remainder Theorem: $f(c)$ equals the remainder when $f(x)$ is divided by $(x - c)$.

AP Precalc Tip: Evaluate $f(c)$ to test if c is a root. Synthetic division evaluates $f(c)$ efficiently and gives the quotient too.

Example: For $f(x) = x^3 - 4x$: $f(2) = 0$ so $x - 2$ is a factor; $f(1) = -3$ is the remainder of $f/(x-1)$.

25 Fundamental Theorem of Algebra

Explanation: A degree n polynomial has exactly n complex roots (counting multiplicity). Real roots are a subset of these.

AP Precalc Tip: A degree n polynomial may have fewer real roots but always n complex roots total. Each root's multiplicity contributes to the total.

Example: $x^4 - 16$ has 4 complex roots: ± 2 and $\pm 2i$.

26 Multiplicity & Behavior at Zeros

Explanation: The multiplicity of a zero is the exponent of its factor. Even multiplicity: the graph touches and turns (does NOT cross). Odd multiplicity: the graph crosses the x-axis.

AP Precalc Tip: A zero with even multiplicity bounces off the axis; odd multiplicity crosses through. This is a frequent AP graph question.

Example: $f(x) = (x-1)^2(x+2)$ crosses at $x = -2$ but just touches at $x = 1$.

27 Polynomial Graphs & Zeros

Explanation: The graph of a polynomial is smooth and continuous (no breaks, corners, or gaps). At each zero, the multiplicity decides cross vs touch.

AP Precalc Tip: Polynomial graphs can never jump or have vertical asymptotes. Use zeros, end behavior, and turning points to sketch any polynomial.

Example: $f(x) = x(x-3)(x+1)$ crosses the axis at -1, 0, and 3 --- three distinct odd-multiplicity zeros.

28 Leading Coefficient Test

Explanation: End behavior is set by the leading term $a \cdot x^n$: even n with $a > 0$ = both up; even n with $a < 0$ = both down; odd n with $a > 0$ = down then up; odd n with $a < 0$ = up then down.

AP Precalc Tip: Always identify the sign and degree of the LEADING term first; every other term is negligible as $|x|$ is large.

Example: $f(x) = -x^5 + 2x$ rises on the left and falls on the right (odd degree, negative lead).

29 Transformations of Functions

Explanation: $y = a \cdot f(b(x - h)) + k$ shifts, stretches, and reflects the base graph. h shifts horizontally (opposite sign), k shifts vertically, a stretches/reflects vertically, b stretches/reflects horizontally.

AP Precalc Tip: A horizontal shift uses $x - h$ (shift right for positive h). The order matters; a negative a or b reflects the graph.

Example: $f(x) = -(x - 2)^3$ shifts x^3 right 2 and reflects over the x-axis.

30 Vertical & Horizontal Shifts

Explanation: $y = f(x) + k$ shifts up ($k > 0$) or down ($k < 0$). $y = f(x - h)$ shifts right ($h > 0$) or left ($h < 0$). The shift moves every point of the graph.

AP Precalc Tip: For a horizontal shift, the sign in the parentheses is opposite the direction. $f(x-3)$ moves the graph RIGHT 3 units.

Example: $g(x) = f(x) - 4$ moves every point of f down 4 units. $h(x) = f(x+1)$ moves f left 1 unit.

31 Vertical & Horizontal Stretches

Explanation: $y = a \cdot f(x)$ stretches vertically by $|a|$ and reflects if $a < 0$. $y = f(b \cdot x)$ compresses horizontally by $1/|b|$ and reflects if $b < 0$.

AP Precalc Tip: A vertical stretch multiplies y-coordinates; a horizontal stretch divides x-coordinates. $a > 1$ stretches up, $|b| > 1$ compresses horizontally.

Example: $y = 2 \sin(x)$ has amplitude 2 (stretched vertically). $y = \sin(2x)$ has period $2\pi/2 = \pi$.

32 Dilations & Reflections Summary

Explanation: Reflecting over the x-axis: $y = -f(x)$. Over the y-axis: $y = f(-x)$. Over the origin: $y = -f(-x)$. Combine with stretches by nesting operations.

AP Precalc Tip: Apply reflections inside (horizontal) with sign flips on x , and outside (vertical) with sign flips on the function value. Test a known point to check.

Example: To reflect f across the y-axis, replace x with $-x$; across the x-axis, multiply by -1 .

33 Composition of Functions

Explanation: $(f \circ g)(x) = f(g(x))$: apply g first, then f to the result. The domain of $f \circ g$ excludes inputs where $g(x)$ is undefined or $g(x)$ is outside the domain of f .

AP Precalc Tip: Composition is NOT generally commutative: $f(g(x)) \neq g(f(x))$. Evaluate inside-out: compute $g(x)$ first, then feed into f .

Example: $f(x) = x^2$, $g(x) = x + 1$: $(f \circ g)(3) = f(4) = 16$. $(g \circ f)(3) = g(9) = 10$ (they differ).

34 Finding a Composite's Value

Explanation: To compute $(f \circ g)(a)$, evaluate $g(a)$ first, then apply f to that result. Trace carefully when functions are given as tables or graphs.

AP Precalc Tip: *Work inside-out and keep track of which output becomes the next input. With graphs, read $g(a)$ off the g graph, then f of that value.*

Example: From tables: if $g(2) = 5$ and $f(5) = 7$, then $(f \circ g)(2) = 7$.

35 Composition & Inverses

Explanation: If f and g are inverses, then $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$ for all x in the domain. Inverses undo each other in either order.

AP Precalc Tip: *Two functions are inverses exactly when their composition returns the input. This is the algebra definition of inverse functions.*

Example: $(f \circ f^{-1})(x) = x$ and $(f^{-1} \circ f)(x) = x$ whenever f is invertible.

36 Inverse Functions

Explanation: The inverse f^{-1} swaps inputs and outputs: if $f(a) = b$ then $f^{-1}(b) = a$. Graphically, inverses are reflections across the line $y = x$. Only one-to-one functions have inverses.

AP Precalc Tip: *To find an inverse algebraically: swap x and y , then solve for y . The domain of f must match the range of f^{-1} and vice versa.*

Example: $f(x) = 2x + 3$: swap to $x = 2y + 3$, solve $y = (x - 3)/2$, so $f^{-1}(x) = (x - 3)/2$.

37 One-to-One & Horizontal Line Test

Explanation: A function is one-to-one if every y has exactly one x (horizontal line test). Only one-to-one functions have inverses.

AP Precalc Tip: *If a horizontal line hits the graph more than once, the function is not one-to-one and has no inverse over that domain.*

Example: $f(x) = x^2$ is not one-to-one on all reals (fails horizontal line test), but is invertible on $x \geq 0$.

38 Rational Functions & Domain

Explanation: A rational function is a ratio $p(x)/q(x)$ of polynomials. It is undefined where the denominator equals zero; those x -values are excluded from the domain.

AP Precalc Tip: *Exclude only the zeros of the denominator. Simplify first --- a common factor may create a hole instead of an asymptote.*

Example: $f(x) = (x+1)/(x^2 - 4)$ is undefined at $x = \pm 2$. Domain: all reals except ± 2 .

39 Vertical Asymptotes

Explanation: A vertical asymptote occurs at $x = c$ where the denominator is zero and the numerator is nonzero after simplifying. As x approaches c , $|f|$ grows without bound.

AP Precalc Tip: *Find vertical asymptotes at denominator zeros that survive simplification. If a zero cancels (common factor), it is a hole, not an asymptote.*

Example: $f(x) = 1/(x-2)$ has a vertical asymptote at $x = 2$. $f(x) = (x-2)/(x-2)(x+1)$ has a hole at $x = 2$ and an asymptote at $x = -1$.

40 Holes (Removable Discontinuities)

Explanation: If p and q share a factor $(x - c)$, the graph has a hole at $x = c$ where the function is undefined but the limit exists. Remove the common factor to find the hole's y -value.

AP Precalc Tip: *A hole is the point where a cancelled factor's zero sits; substitute into the simplified function to find the missing y -value.*

Example: $f(x) = (x^2 - 1)/(x - 1) = x + 1$ after cancelling, with a hole at $(1, 2)$.

41 Horizontal Asymptotes

Explanation: Compare degrees: if degree of numerator $<$ denominator, $y = 0$. If equal, $y =$ leading coefficient ratio. If numerator degree is larger, there is no horizontal asymptote (oblique possible).

AP Precalc Tip: *The three cases are decided by comparing the degrees of numerator and denominator. Only the leading coefficients matter in the equal case.*

Example: $f(x) = (3x^2 + 1)/(2x^2 - 5)$ has horizontal asymptote $y = 3/2$. $f(x) = x/(x^2 + 1)$ has $y = 0$.

42 Oblique (Slant) Asymptotes

Explanation: When the numerator's degree is exactly one more than the denominator's, there is a slant asymptote given by the quotient of polynomial long division.

AP Precalc Tip: Divide the numerator by the denominator; the quotient (ignoring the remainder) is the slant asymptote line.

Example: $f(x) = (x^2 + 1)/(x - 1)$: long division gives $x + 1$ with remainder 2, so the slant asymptote is $y = x + 1$.

43 Zeros of Rational Functions

Explanation: The x-intercepts (zeros) of a rational function are where the NUMERATOR is zero, provided that x is in the domain (not cancelled and not an asymptote).

AP Precalc Tip: Find zeros from the numerator, but exclude any x that also zeroes the denominator (hole). The graph crosses the x-axis at valid numerator zeros.

Example: $f(x) = (x-3)/(x+2)$ has a zero at $x = 3$ (and a vertical asymptote at $x = -2$).

44 Sign Analysis of Rational Functions

Explanation: The sign of a rational function changes only at its zeros and vertical asymptotes. Test one point per interval to build a sign chart and sketch the graph.

AP Precalc Tip: Mark zeros and asymptotes on a number line, then pick a test value in each interval to determine the sign. This powers accurate sketching.

Example: For $f(x) = (x-1)/(x+1)$, test $x = -2$ (positive), $x = 0$ (negative), $x = 2$ (positive).

45 Graphing Rational Functions

Explanation: Sketch in order: find zeros, vertical asymptotes, horizontal/oblique asymptote, and intercepts; then use sign analysis to connect the pieces.

AP Precalc Tip: A systematic order prevents missing a branch. Always include both the asymptote lines (dashed) and the behavior as x approaches them.

Example: $f(x) = (2x)/(x^2 - 1)$ has a zero at 0, vertical asymptotes at ± 1 , and horizontal asymptote $y = 0$.

46 Rational Inequalities

Explanation: Solve $p(x)/q(x) \geq 0$ by finding zeros and vertical asymptotes, testing intervals, and selecting those where the sign matches. Never include asymptote x-values in the solution.

AP Precalc Tip: The sign can change at zeros and at undefined points. Remember to EXCLUDE values that make the denominator zero even if the inequality has $\leq$.

Example: Solve $(x+1)/(x-2) \geq 0$: the sign is nonnegative on $(-\infty, -1]$ and $(2, \infty)$.

47 Modeling with Polynomials & Rational Functions

Explanation: Polynomials model smooth growth; rational functions model rates like average cost and concentration that approach a limiting value.

AP Precalc Tip: Read the context: a horizontal asymptote often represents a limiting quantity (e.g., carrying capacity, long-run cost). Zeros represent break-even points.

Example: Average cost $C(x) = (\text{fixed} + \text{variable})/x$ approaches the variable unit cost as x grows --- a horizontal asymptote.

48 Polynomial Degree & Leading Term

Explanation: The degree is the highest exponent; the leading term is $a \cdot x^n$ where a is the leading coefficient. The degree limits turning points and roots.

AP Precalc Tip: The degree tells you the maximum number of x-intercepts (n) and turning points (n-1). State the degree from the highest power.

Example: $f(x) = 4x^5 - 2x^2 + 1$ has degree 5, leading term $4x^5$, up to 5 real roots.

49 Constant & Linear Differences

Explanation: For equally spaced x, linear functions have constant first differences and polynomials of degree k have constant kth differences.

AP Precalc Tip: Recognizing difference patterns identifies the degree of a polynomial from data --- this appears in AP data-analysis questions.

Example: Quadratic data (x:0,1,2,3) have constant second differences; cubic data have constant third differences.

50 Roots, Zeros, x-Intercepts

Explanation: A root (zero) of $f(x)$ is an x where $f(x) = 0$ --- where the graph crosses or touches the x -axis. The real roots are the x -intercepts.

AP Precalc Tip: Solve $f(x) = 0$ to find x -intercepts. A root with even multiplicity touches; odd multiplicity crosses. Count roots up to the degree.

Example: $f(x) = x^3 - 4x = x(x-2)(x+2)$ has zeros at -2 , 0 , and 2 .

51 The Remainder & Factor Theorems Applied

Explanation: Use synthetic division to evaluate a polynomial at c and to test if $(x - c)$ is a factor (remainder 0). This is the fastest tool for higher-degree polynomials.

AP Precalc Tip: Synthetic division gives both the quotient coefficients and the remainder $f(c)$. A zero remainder means c is a root.

Example: Dividing $x^3 - 1$ by $(x - 1)$ via synthetic division gives remainder 0 , so $x = 1$ is a root.

52 Graph Transformations of Polynomials

Explanation: Shifting $f(x)$ to $f(x - h) + k$, reflecting $-f(x)$, and stretching $a \cdot f(x)$ apply to polynomials like any function.

AP Precalc Tip: Apply transformations in order: stretch, reflect, shift. Track how the y -intercept and any vertex move.

Example: $g(x) = (x - 2)^3 - 1$ shifts x^3 right 2 and down 1 .

53 Rational Function End Behavior

Explanation: Long-run behavior of rational functions is set by comparing leading terms, giving horizontal or oblique asymptotes as x gets large.

AP Precalc Tip: Divide leading terms to find the horizontal asymptote. If degrees differ by one, long division gives the oblique asymptote.

Example: $f(x) = (2x^2 + 1)/(x^2 - 4)$ approaches 2 as $|x|$ grows: horizontal asymptote $y = 2$.

54 Rational Zeros & Intercepts

Explanation: The x -intercepts of a rational function come from numerator zeros (that survive simplification); the y -intercept is $f(0)$.

AP Precalc Tip: Set the numerator to zero for x -intercepts; evaluate $f(0)$ for the y -intercept. Ignore canceled factors (holes).

Example: $f(x) = (x-1)(x+2)/(x-3)$: x -intercepts at 1 and -2 ; y -intercept at $f(0) = (-2)/(-3) = 2/3$.

55 Polynomial Inequalities

Explanation: Solve $f(x) > 0$ by finding roots and testing sign in each interval between them. Roots where multiplicity is even do not change signs.

AP Precalc Tip: Build a sign chart from the roots. Even-multiplicity roots do not flip the sign; odd ones do. Include/exclude endpoints by the inequality.

Example: For $(x-1)(x-3) > 0$, the solution is $x < 1$ or $x > 3$ (outside the roots).

56 Modeling Polynomials from Data

Explanation: Fit a polynomial of appropriate degree to data by identifying difference patterns or using regression, then interpret in context.

AP Precalc Tip: Match the degree to the data's curvature/differences. Use the model to interpolate within the data range.

Example: Revenue data with constant third differences are modeled well by a cubic polynomial.

57 Rational Function Vertical/Horizontal Behavior in Context

Explanation: In applied problems, vertical asymptotes represent undefined/boundary points and horizontal asymptotes represent limiting steady-state values.

AP Precalc Tip: Connect each asymptote to a real meaning: max capacity, breaking point, or long-run limit. Interpret, not just state.

Example: The number of items a factory can produce approaches a capacity limit --- a horizontal asymptote in the model.

Unit 2: Exponential and Logarithmic Functions

27-40% of the exam. Exponential growth/decay, logarithms, and their applications.

58 Exponential Functions

Explanation: $f(x) = a \cdot b^x$ with a = initial value, b = base. If $b > 1$ it grows; if $0 < b < 1$ it decays. Base $0 < b$, $b \neq 1$, and $a \neq 0$.

AP Precalc Tip: Growth/decay is determined by whether the base is greater or less than 1. b^x is ALWAYS positive for $b > 0$, so exponential functions never output zero.

Example: $f(x) = 100(1.05)^t$ grows 5% per period. $g(x) = 50(0.8)^t$ decays 20% per period.

59 Properties of Exponents

Explanation: $a^m \cdot a^n = a^{(m+n)}$; $(a^m)^n = a^{(m \cdot n)}$; $a^m / a^n = a^{(m-n)}$; $a^0 = 1$; $a^{-n} = 1/a^n$; $(ab)^n = a^n b^n$. These rules simplify exponential expressions.

AP Precalc Tip: Combine bases only when they match. A negative exponent means reciprocal, not a negative number. $a^0 = 1$ for any nonzero a .

Example: $x^3 \cdot x^5 = x^8$. $(x^2)^4 = x^8$. $2^{-3} = 1/8$.

60 Exponential Growth Model

Explanation: Model $y = a \cdot b^x$ where $b = 1 + r$ and r is the growth rate (as a decimal). The quantity increases by a fixed percentage each period.

AP Precalc Tip: A 6% growth rate means $b = 1.06$, not 0.06. Each unit step multiplies the amount by b . Look for the constant multiplier in data tables.

Example: If a population grows 4% per year from 2000, the function is $P(t) = 2000(1.04)^t$.

61 Exponential Decay Model

Explanation: Model $y = a \cdot b^x$ where $b = 1 - r$ is the decay factor. The quantity decreases by a fixed percentage each period.

AP Precalc Tip: A 30% decay means $b = 0.70$. Decay graphs approach 0 but never reach it. Use the same form as growth with the base between 0 and 1.

Example: A drug decays 25% per hour from 80 mg: $A(t) = 80(0.75)^t$.

62 Percentage Change & Base

Explanation: A base of b corresponds to a change of $(b - 1) \cdot 100\%$ per step. Negative change = decay; this connects percentages to exponential bases.

AP Precalc Tip: Convert between form: growth rate $r \rightarrow$ base $b = 1 + r$. Be careful: a 50% increase ($b=1.5$) is not the same as doubling ($b=2$).

Example: Base 1.12 means +12% per step; base 0.9 means -10% per step.

63 Compound Interest

Explanation: $A = P(1 + r/n)^{(n \cdot t)}$ where P = principal, r = annual rate, n = compounds per year, t = years. As n grows, this approaches continuous compounding.

AP Precalc Tip: Compound interest exponentials grow with base $(1 + r/n)$. More frequent compounding gives a slightly larger amount; n matters via the exponent $n \cdot t$.

Example: $P = 1000$, $r = 0.05$, $n = 12$, $t = 2$: $A = 1000(1 + 0.05/12)^{(24)}$.

64 The Number e

Explanation: e is approximately 2.718 and is the natural base. Continuous growth/decay uses $A = P \cdot e^{(rt)}$. $e^{(rt)}$ is the limit of $(1 + r/n)^{(n \cdot t)}$ as n goes to infinity.

AP Precalc Tip: $e^{(rt)}$ models continuous change. On the AP exam, e is the default base for natural exponential models; know $e \approx 2.718$.

Example: Continuous growth of 5% on 1000 for 3 years: $A = 1000 \cdot e^{(0.05 \cdot 3)}$.

65 Continuous Growth & Decay

Explanation: Model $A(t) = A_0 e^{k(t)}$: if $k > 0$ it grows, if $k < 0$ it decays. k is the continuous rate, not the same as the periodic percentage rate.

AP Precalc Tip: A continuous rate of -0.02 is a 2% continuous decay. Convert periodic to continuous with $k = \ln(1 + r)$.

Example: Population with $k = 0.02$ per year: $P(t) = P_0 e^{(0.02 t)}$.

66 Exponential Functions in Tables

Explanation: If a table has equal x -steps, exponential data show a CONSTANT RATIO between successive y -values. Linear data show constant differences.

AP Precalc Tip: Divide consecutive y -values: if all ratios are equal, the function is exponential with that base. This is the key data-recognition skill.

Example: y : 2, 6, 18, 54 --- ratios all 3, so it is exponential with base 3: $y = 2 \cdot 3^x$ (for integer x).

67 Doubling Time & Half-Life

Explanation: Doubling time is the period to multiply a quantity by 2; half-life is the period to halve it. Both are constant for exponential functions.

AP Precalc Tip: Because exponential functions have constant ratios, doubling time and half-life are fixed regardless of the starting amount.

Example: A substance with half-life 10 days: after 30 days only $(1/2)^3 = 1/8$ remains.

68 Solving Exponential Equations (Same Base)

Explanation: If both sides can be written with the same base, set the exponents equal: $b^u = b^v$ implies $u = v$. This is the fastest method when applicable.

AP Precalc Tip: Rewrite numbers like 8 and 32 as powers of 2 before comparing exponents. This method works only when you can match bases.

Example: $2^x = 32 = 2^5$, so $x = 5$. $16^x = 4$: $2^{(4x)} = 2^2$, so $4x = 2$, $x = 1/2$.

69 Introduction to Logarithms

Explanation: $y = \log_b(x)$ asks: to what power y must b be raised to equal x ? Equivalently, $b^y = x$ means $\log_b(x) = y$. Logs are inverses of exponentials.

AP Precalc Tip: Translate between forms: $\log_b(x) = y \iff b^y = x$. The base b must be positive (not 1) and $x > 0$.

Example: $\log_2(8) = 3$ because $2^3 = 8$. $\log_{10}(1000) = 3$ because $10^3 = 1000$.

70 Common & Natural Logarithms

Explanation: Common log is base 10, written $\log(x)$ (or $\log_{10}$). Natural log is base e , written $\ln(x)$ (or $\log_e$). Your calculator has both.

AP Precalc Tip: On most calculators $\log$ means base 10 and $\ln$ means base e . $\ln(e) = 1$ and $\ln(1) = 0$ --- memorize these useful anchors.

Example: $\ln(e^2) = 2$, $\log(100) = 2$, $\ln(1) = 0$.

71 Logarithm Properties

Explanation: $\log_b(xy) = \log_b(x) + \log_b(y)$; $\log_b(x/y) = \log_b(x) - \log_b(y)$; $\log_b(x^p) = p \log_b(x)$. These expand and condense logarithmic expressions.

AP Precalc Tip: These mirror the exponent rules. They let you expand a single log or condense a sum/difference into one log (needed for solving).

Example: $\log(x^2 y) = 2 \log(x) + \log(y)$. $\log(10x/3) = \log(10x) - \log(3)$.

72 Change of Base Formula

Explanation: $\log_b(a) = \ln(a)/\ln(b) = \log(a)/\log(b)$. Use it to evaluate logs in any base with a calculator or to rewrite a log.

AP Precalc Tip: Change of base lets you compute a log with any base using $\log$ or $\ln$. It also converts between bases in algebra problems.

Example: $\log_2(10) = \ln(10)/\ln(2) \sim 3.322$. To find $\log_3(5)$, use $\ln(5)/\ln(3)$.

73 Unit Circle / Log Graphs Basics

Explanation: The graphs of $y = b^x$ and $y = \log_b(x)$ are reflections across $y = x$. The exponential passes through (0,1); the log through (1,0), with a vertical asymptote at $x = 0$.

AP Precalc Tip: A log graph only exists for $x > 0$ and rises slowly; it has a vertical asymptote at the y-axis. The exponential has a horizontal asymptote at $y = 0$.

Example: $y = 2^x$ passes (0,1); its inverse $y = \log_2(x)$ passes (1,0) with a vertical asymptote at $x = 0$.

74 Graphing Exponential Functions

Explanation: An exponential has a horizontal asymptote (typically $y = 0$), passes through (0, a), and is always positive. It is one-to-one and increases or decreases monotonically.

AP Precalc Tip: The horizontal asymptote tells the long-run level. With a vertical shift ($y = b^x + k$), the asymptote moves to $y = k$.

Example: $f(x) = 3^x + 2$ has horizontal asymptote $y = 2$ and passes through (0, 3).

75 Graphing Logarithmic Functions

Explanation: $y = \log_b(x)$ has a vertical asymptote at $x = 0$, passes through (1, 0) and (b, 1), and is increasing ($b > 1$). Its domain is $x > 0$.

AP Precalc Tip: The vertical asymptote is at $x = 0$ (or $x = h$ after a horizontal shift). Pick a few $x = b^n$ points to sketch the curve.

Example: $y = \log_2(x)$ passes through (1,0) and (2,1); it rises slowly and approaches the y-axis.

76 Transforming Exponential/Log Graphs

Explanation: Shifts and stretches apply as usual: $y = a \cdot b^{(x-h)} + k$ moves the horizontal asymptote to k and shifts horizontally by h ; $y = \log_b(x+c)$ shifts the vertical asymptote left/right.

AP Precalc Tip: The asymptote follows vertical shifts exactly. Set the inside equal to zero to find the new vertical asymptote of a shifted log.

Example: $y = 2^{(x-1)} - 3$ has horizontal asymptote $y = -3$. $y = \log_2(x-2)$ has vertical asymptote $x = 2$.

77 Modeling Exponential Fit

Explanation: To fit an exponential model to two points, use the base from the ratio and solve for the initial value a . General form: solve two equations in a and b .

AP Precalc Tip: Given two points $(x_1, y_1), (x_2, y_2)$: $b = (y_2/y_1)^{1/(x_2-x_1)}$ after dividing equations, then solve for a .

Example: Through (0,5) and (2,20): $b^2 = 4$ so $b = 2$, $a = 5$: model $y = 5(2)^x$.

78 Solving Exponential Equations (Logs)

Explanation: When bases differ, take $\ln$ or $\log$ of both sides and use the exponent property to bring the exponent down: solve for the variable.

AP Precalc Tip: Take the natural log of both sides to extract exponents. Isolate the term with the variable first, then apply logs.

Example: Solve $3^x = 20$: $\ln(3^x) = \ln(20)$, $x \ln(3) = \ln(20)$, $x = \ln(20)/\ln(3)$.

79 Solving Logarithmic Equations

Explanation: Isolate the log, then rewrite in exponential form, or combine logs with log properties and convert. Always check that solutions are in the domain ($\arg > 0$).

AP Precalc Tip: After solving, plug back in to make sure no argument is negative or zero --- extraneous solutions are common with logs.

Example: $\log_2(x) + \log_2(x-2) = 3 \rightarrow \log_2(x(x-2)) = 3 \rightarrow x(x-2) = 8 \rightarrow x = 4$ ($x = -2$ rejected).

80 Extraneous Solutions in Logs

Explanation: A solution that makes a log argument nonpositive is extraneous and must be discarded, even if it satisfies the algebra after converting.

AP Precalc Tip: Always check the ORIGINAL equation's domain (each log argument > 0). Reject any candidate that violates it.

Example: Solving $\log(x-5) + \log(x) = \log(6)$ may give $x = 6$ and $x = -1$; -1 is extraneous because $\log(-1)$ is undefined.

81 Exponential Regression & Interpolation

Explanation: Many real datasets follow $y = a \cdot b^x$; use regression or two points to estimate parameters, then interpolate or extrapolate values.

AP Precalc Tip: Distinguish interpolation (inside the data range, reliable) from extrapolation (outside, less reliable). AP asks which values are reasonable predictions.

Example: If a model fits points (0,10) and (3,80), predicting y at $t = 1$ is interpolation; at $t = 50$ it is extrapolation.

82 Doubling Time Formula

Explanation: For a quantity doubling every d periods: $y = a \cdot 2^{(t/d)}$. The exponent t/d counts doublings. Solve for d using any known value.

AP Precalc Tip: The exponent is the number of doubling periods, not time alone. This form is efficient when doubling time is given.

Example: Doubling every 3 years: $y = a \cdot 2^{(t/3)}$. After 6 years it has doubled twice: factor 4.

83 Logistic Growth (Intro)

Explanation: Logistic growth starts nearly exponential but approaches a carrying capacity L : $P(t) = L / (1 + c \cdot e^{(-k t)})$. Growth slows near L .

AP Precalc Tip: The logistic model has a horizontal asymptote at the carrying capacity L --- a population can't grow forever. AP contrasts this with pure exponential growth.

Example: A fish population growing logistically approaches a carrying capacity of 5000 fish rather than growing without bound.

84 Logarithms as Inverses

Explanation: Because logs and exponentials are inverses: $\log_b(b^x) = x$ and $b^{(\log_b(x))} = x$. These identities let you simplify expressions.

AP Precalc Tip: log and exponential cancel each other when the bases match and the composition is in the right order. This is the key to simplifying.

Example: $\ln(e^7) = 7$. $e^{(\ln(5))} = 5$. $\log(10^{100}) = 100$.

85 Domain of Log Functions

Explanation: The argument of a log must be positive: $\log_b(g(x))$ requires $g(x) > 0$. Solve the inequality to find the domain.

AP Precalc Tip: Set the inside greater than zero and solve. The domain is intervals of x where the argument is positive.

Example: $f(x) = \log_2(x - 4)$: domain $x > 4$, vertical asymptote at $x = 4$.

86 Applying Exponential Models

Explanation: Exponential models apply to population, finance, radioactivity, drug dosage, cooling (Newton's law), and depreciation. Identify the model from the context words.

AP Precalc Tip: Look for key phrases: 'doubles/halves every period' (base 2 or 1/2), 'grows/decays by $r\%$ ' (base $1+r$ or $1-r$), 'continuously' (e).

Example: Newton's cooling $T(t) = T_{\text{env}} + (T_0 - T_{\text{env}}) \cdot e^{(-k t)}$ approaches room temperature asymptotically.

87 Comparison of Growth Models

Explanation: Linear grows by a constant amount; quadratic by increasing amounts; exponential by a constant percentage (eventually fastest); logistic approaches a cap.

AP Precalc Tip: For large t , exponential ($b > 1$) outgrows all polynomial growth. If a context limits growth, use logistic; otherwise exponential.

Example: Over many steps, exponential will overtake linear and quadratic even if it starts much smaller.

88 Piecewise Exponential Contexts

Explanation: Some models change rules across intervals (e.g., different growth rates in different years). Evaluate with the matching piece.

AP Precalc Tip: Identify which piece applies to the given time t , then evaluate that formula. Watch the breakpoint: does the model switch at $t = c$ exactly?

Example: A loan might grow 3% the first 2 years then 5% after: use the 3% piece for $t < 2$, the 5% piece for $t \geq 2$.

89 Inverse of Exponential = Log

Explanation: Since logs invert exponentials, solving for an exponent (time) in a growth model requires logs: given $y = a b^t$, solve $t = \log_b(y/a)$.

AP Precalc Tip: To find when a quantity reaches a target, isolate the base-power and take the log of both sides. This is the most tested application.

Example: Find when $100(1.1)^t = 200$: $(1.1)^t = 2$, $t = \ln(2)/\ln(1.1)$.

90 Exponent Rules with Variable Bases

Explanation: Manipulate expressions like $(a^x)/(a^y) = a^{(x-y)}$ and $(a^x)^y = a^{(xy)}$. Combine only like bases; factor common powers.

AP Precalc Tip: When simplifying, keep one base per term and combine exponents correctly. Watch for negatives and fractions in exponents (roots).

Example: $x^{(1/2)} * x^{(3/2)} = x^2$. $(a^2 b^3)/(a b) = a b^2$. $x^{(1/2)} = \sqrt{x}$.

91 Graph Interpretation of Asymptotes

Explanation: On an exponential graph, the horizontal asymptote is the level the function approaches; on a log graph, the vertical asymptote is where it approaches $\pm$ infinity.

AP Precalc Tip: A graph approaching a horizontal line $y = c$ means the quantity tends to c . This connects graphs to equations and real limits.

Example: A cooling cup of coffee approachng 20 degrees is a horizontal asymptote $y = 20$.

92 Exponential Regression & Interpolation

Explanation: Many real datasets follow $y = a b^x$; use regression or two points to estimate parameters, then interpolate or extrapolate values.

AP Precalc Tip: Distinguish interpolation (inside the data range, reliable) from extrapolation (outside, less reliable). AP asks which values are reasonable predictions.

Example: If a model fits points (0,10) and (3,80), predicting y at $t = 1$ is interpolation; at $t = 50$ it is extrapolation.

93 Combining Exponent Rules

Explanation: Apply multiple exponent rules together: distribute powers over products/quotients and simplify variable exponents into a single base-power form.

AP Precalc Tip: Simplify step by step, keeping one base per term. Watch nested exponents: $(a^m)^n = a^{(mn)}$, and $a^m * a^n = a^{(m+n)}$.

Example: $(2x^3)^2 / (x^2) = 4x^6 / x^2 = 4x^4$.

94 Solving for the Base or Exponent

Explanation: To solve $b^x = y$, take $\log_b$ of both sides ($x = \log_b y$), or use change of base. To find b , raise both sides to $1/x$.

AP Precalc Tip: Using logs isolates the exponent; using roots ($1/x$ power) isolates the base. Pick the operation that undoes what is applied to the variable.

Example: Given $y = 8 * 2^x$, solving $32 = 8 * 2^x$ gives $2^x = 4$, $x = 2$. Given $100 = a * 5^2$, $a = 4$.

95 Logarithm of Exponential & Vice Versa

Explanation: The inverse identities $\log_b(b^x) = x$ and $b^{(\log_b x)} = x$ let you swap log and exponential forms cleanly.

AP Precalc Tip: Match the base: the log base must equal the exponential base for these identities. Use them to simplify nested expressions.

Example: $\log_3(3^{10}) = 10$. $10^{(\log(7))} = 7$. $e^{(\ln(2x))} = 2x$.

96 Semi-Log & Data Modeling

Explanation: Exponential data plot as a straight line on a semi-log graph (log of y vs x). This linearizes growth for analysis.

AP Precalc Tip: If $\log(y)$ versus x is linear, the data are exponential. This concept appears in advanced data questions.

Example: If points $(x, \log y)$ are collinear, then y is of the form $a * b^x$.

97 Population & Doubling Application

Explanation: Exponential models predict population growth: $P = P_0 \cdot b^t$ or with doubling time $P = P_0 \cdot 2^{(t/d)}$.

AP Precalc Tip: Match the model to the wording: percentage growth $\rightarrow$ base $1+r$; doubling $\rightarrow$ base 2 with exponent t/d .

Example: A population doubling every 8 years: $P(t) = 1000 \cdot 2^{(t/8)}$. After 24 years: 8000.

98 Compound vs Continuous Comparison

Explanation: As compounding frequency increases, $A = P(1 + r/n)^{nt}$ approaches $P \cdot e^{(rt)}$ (continuous). The difference is small at low rates.

AP Precalc Tip: Recognize when to use discrete vs continuous compounding from the problem. Both use the same P , r , t meanings.

Example: At 6% for 10 years on 1000: annual gives ~ 1790.8 ; continuous gives ~ 1822.1 (larger).

99 Interpreting Exponential Model Parameters

Explanation: In $y = a \cdot b^x$, a is the starting value ($x = 0$) and b is the multiplier per unit step. Identify each from context.

AP Precalc Tip: Pull a and b from the problem: a = initial amount, b = per-period factor. Convert b to a percent change.

Example: $y = 500(0.92)^t$ starts at 500 and decays 8% per period because $1 - 0.92 = 0.08$.

100 Exponential Functions in Science Contexts

Explanation: Radioactive decay, drug elimination, and bacterial growth are natural exponential contexts with real constants.

AP Precalc Tip: Recognize the context and map it: radioactive $\rightarrow$ half-life; drug $\rightarrow$ elimination rate; bacteria $\rightarrow$ growth rate.

Example: A substance with half-life T has amount $A(t) = A_0 \cdot (1/2)^{(t/T)}$.

Unit 3: Trigonometric and Polar Functions

30-35% of the exam. Trig functions, unit circle, identities, and polar coordinates.

101 Angles & Radian Measure

Explanation: One radian is the angle subtended by an arc equal to the radius. Full circle = 2π radians = 360 degrees. Convert with π radians = 180 degrees.

AP Precalc Tip: Convert degrees to radians by multiplying by $\pi/180$; radians to degrees by multiplying by $180/\pi$. Radians are the default on the AP exam.

Example: 90 degrees = $\pi/2$. 60 degrees = $\pi/3$. π radians = 180 degrees.

102 Degree-Radian Conversions

Explanation: Use the ratio π rad = 180 deg. Common angles to memorize: 30 = $\pi/6$, 45 = $\pi/4$, 60 = $\pi/3$, 90 = $\pi/2$, 180 = π .

AP Precalc Tip: Quickly re-derive unknown conversions from the anchor π rad = 180 deg. Most trig questions on AP use radians.

Example: 45 degrees converts to $\pi/4$. $3\pi/2$ converts back to 270 degrees.

103 Arc Length & Sector Area

Explanation: Arc length $s = r\theta$ (θ in radians). Sector area = $(1/2)r^2\theta$ (θ in radians).

AP Precalc Tip: These formulas require θ in RADIANS. Convert first if the angle is in degrees.

Example: A circle $r = 5$ with $\theta = \pi/3$ has arc length $5(\pi/3) = 5\pi/3$.

104 Unit Circle: Sine & Cosine

Explanation: On the unit circle, $\cos(\theta)$ is the x-coordinate and $\sin(\theta)$ is the y-coordinate of the point reached by rotating θ from the positive x-axis.

AP Precalc Tip: $\cos = x$, $\sin = y$ on the unit circle. This is the foundation for all trig values and the graphs.

Example: At $\theta = \pi/2$, the point is (0,1), so $\cos(\pi/2) = 0$ and $\sin(\pi/2) = 1$.

105 Common Unit Circle Values

Explanation: Memorize: $\sin/\cos$ of 0, $\pi/6$, $\pi/4$, $\pi/3$, $\pi/2$ and their quadrant sign patterns. $\sin(\pi/6) = 1/2$, $\cos(\pi/3) = 1/2$, etc.

AP Precalc Tip: The special angles are tested constantly. Build the unit circle table and know signs by quadrant: All Students Take Calculus (all, sin, tan, cos).

Example: $\cos(\pi/3) = 1/2$, $\sin(\pi/4) = \sqrt{2}/2 \approx 0.707$, $\sin(0) = 0$.

106 Tangent & Reciprocal Trig

Explanation: $\tan(\theta) = \sin/\cos$. The reciprocals: $\csc = 1/\sin$, $\sec = 1/\cos$, $\cot = 1/\tan = \cos/\sin$. They are undefined where their denominator is zero.

AP Precalc Tip: $\tan$ is undefined where $\cos = 0$ (at $\pi/2$, $3\pi/2$...). $\sec$ and $\csc$ are undefined where $\cos$ or $\sin = 0$ respectively.

Example: $\tan(\pi/4) = 1$. $\sec(0) = 1$. $\csc(\pi/2) = 1$.

107 Reference Angles

Explanation: The reference angle is the acute angle a given angle makes with the x-axis. Use it to find trig values in any quadrant by applying the sign.

AP Precalc Tip: Find the reference angle by rerouting to the nearest x-axis, then use the known special-angle value and adjust the sign by quadrant.

Example: For 150 degrees, the reference angle is 30 degrees; $\sin(150) = +\sin(30) = 1/2$ because sine is positive in quadrant II.

108 Quadrant Signs

Explanation: Quadrant I: all trig positive. Q II: $\sin$ (and $\csc$) positive. Q III: $\tan$ (and $\cot$) positive. Q IV: $\cos$ (and $\sec$) positive.

AP Precalc Tip: Use ASTC to recall which function is positive in each quadrant. Sign errors are a top cause of wrong trig answers.

Example: In quadrant II, $\cos(120) = -\cos(60) = -1/2$ (cosine negative in QII).

109 Trig Function Graphs: Sine & Cosine

Explanation: $y = \sin(x)$ and $y = \cos(x)$ have period 2π , amplitude 1, and range $[-1, 1]$. Cosine is a sine shifted left by $\pi/2$.

AP Precalc Tip: *Sine starts at 0, cosine at 1. Their shape (wave) repeats every 2π ; both oscillate between -1 and 1.*

Example: $\sin(x)$ at $x = 0, \pi/2, \pi, 3\pi/2, 2\pi$ gives 0, 1, 0, -1, 0. Cosine gives 1, 0, -1, 0, 1.

110 Graphing $y = a \sin(bx)$ (amplitude & period)

Explanation: For $y = a \sin(bx)$: amplitude = $|a|$, period = $2\pi/|b|$, and the frequency is $|b|/(2\pi)$. Larger $|a|$ stretches taller; larger $|b|$ compresses horizontally.

AP Precalc Tip: *Amplitude is half the vertical height (peak to trough). Period = $2\pi/b$ is the horizontal length of one full cycle.*

Example: $y = 3 \sin(2x)$ has amplitude 3 and period $2\pi/2 = \pi$.

111 Phase Shift (Horizontal Shift)

Explanation: $y = a \sin(bx - c) + d$ has phase shift c/b and is the graph shifted right by c/b (for $-c$). The midline is $y = d$.

AP Precalc Tip: *The phase shift is c/b (with the standard form). d sets the midline (vertical shift). Order: shift, then graph one period.*

Example: $y = \sin(x - \pi/2)$ is $\sin$ shifted right by $\pi/2$. $y = \sin(x) + 2$ has midline $y = 2$.

112 Midline, Amplitude, Period from a Graph

Explanation: Given a sine/cosine graph: amplitude = $(\max - \min)/2$, midline = $(\max + \min)/2$, period = distance for one full cycle, phase = horizontal offset.

AP Precalc Tip: *Extract these four from the graph to build the equation $y = a \sin(b(x-h)) + d$. The midline is the average height.*

Example: If $\max = 5$ and $\min = 1$, amplitude = 2 and midline $y = 3$. A peak-to-peak distance of 4 gives period 4.

113 Modeling Periodic Behavior

Explanation: $y = a \sin(b(x - h)) + d$ models periodic data: d = midline, a = amplitude, period = $2\pi/b$, h = horizontal shift. Match the model to the context.

AP Precalc Tip: *The midline is the average value (e.g., average temperature), amplitude the swing from average, period the time for full cycle (e.g., 24 h, 365 d).*

Example: Daily temperature alternating between 70 and 84 F: midline 77, amplitude 7, period 24 h.

114 The Tangent Graph

Explanation: $y = \tan(x)$ has period π (not 2π), passes through the origin, and has vertical asymptotes at $\pi/2 + k\pi$ where $\cos(x) = 0$.

AP Precalc Tip: *Tangent repeats every π and has vertical asymptotes where cosine is zero. It rises through each branch from $-\infty$ to $+\infty$.*

Example: $\tan(x)$ has asymptotes at $x = \pi/2$ and $x = 3\pi/2$; between them it passes through $(0,0)$.

115 Inverse Trig Functions

Explanation: $\arcsin/\arccos/\arctan$ (also $\sin^{-1}$ etc.) return the angle whose trig value is given, restricted to principal ranges: $\arcsin$ on $[-\pi/2, \pi/2]$, $\arccos$ on $[0, \pi]$, $\arctan$ on $(-\pi/2, \pi/2)$.

AP Precalc Tip: *Inverse trig gives the PRINCIPAL angle in its restricted range. $\arcsin(\sin(x)) = x$ only when x is in the range; otherwise it returns the equivalent angle.*

Example: $\arcsin(1/2) = \pi/6$. $\arccos(-1) = \pi$. $\arctan(1) = \pi/4$.

116 Solving Basic Trig Equations

Explanation: Solve $\sin(x) = k$ by finding all angles with that sine on one period, then add the period $2\pi \cdot n$ for general solutions (or constrain to the given interval).

AP Precalc Tip: *Use the unit circle for special values; use inverse trig plus symmetry for others. Include ALL solutions in the specified interval.*

Example: $\sin(x) = 1/2$ on $[0, 2\pi]$: $x = \pi/6$ and $x = 5\pi/6$.

117 Pythagorean Identity

Explanation: $\sin^2(x) + \cos^2(x) = 1$. Derived from the unit circle ($x^2 + y^2 = 1$). It is the most-used trig identity.

AP Precalc Tip: Rearrange: $\sin^2 = 1 - \cos^2$, or $\cos^2 = 1 - \sin^2$. Use it to convert between sin and cos or to simplify expressions.

Example: If $\sin(\theta) = 3/5$, then $\cos^2 = 1 - 9/25 = 16/25$, so $\cos(\theta) = \pm 4/5$ (sign by quadrant).

118 Other Pythagorean Identities

Explanation: $\tan^2(x) + 1 = \sec^2(x)$ and $1 + \cot^2(x) = \csc^2(x)$. These follow from dividing the main identity by $\cos^2$ or $\sin^2$.

AP Precalc Tip: Use $\tan^2 + 1 = \sec^2$ to replace sec or tan terms; $1 + \cot^2 = \csc^2$ similarly. Needed frequently for simplification.

Example: $\sec^2(x) - \tan^2(x) = 1$. $\csc^2(x) - \cot^2(x) = 1$.

119 Double-Angle Identities

Explanation: $\sin(2x) = 2 \sin(x) \cos(x)$. $\cos(2x) = \cos^2 - \sin^2 = 1 - 2 \sin^2 = 2 \cos^2 - 1$. $\tan(2x) = 2 \tan(x) / (1 - \tan^2(x))$.

AP Precalc Tip: Pick the $\cos(2x)$ form that simplifies best (matching what you have for sin or cos). $\sin(2x) = 2 \sin \cos$ is the key one.

Example: $\sin(2x) = 2 \sin(x) \cos(x)$. $\cos(2x) = 2 \cos^2(x) - 1$ lets you write $\cos(2x)$ in terms of cos only.

120 Sum & Difference Identities

Explanation: $\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$. $\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$. $\tan(a \pm b) = (\tan a \pm \tan b) / (1 \mp \tan a \tan b)$.

AP Precalc Tip: Know the exact sign patterns: sine changes with the middle sign, cosine flips. Use them to evaluate trig of sums of special angles.

Example: $\sin(75) = \sin(45+30) = \sin 45 \cos 30 + \cos 45 \sin 30 = (\sqrt{2}/2)(\sqrt{3}/2) + (\sqrt{2}/2)(1/2)$.

121 Even-Odd & Cofunction Identities

Explanation: $\sin(-x) = -\sin(x)$ (odd), $\cos(-x) = \cos(x)$ (even), $\tan(-x) = -\tan(x)$. Cofunctions: $\sin(x) = \cos(\pi/2 - x)$, $\tan(x) = \cot(\pi/2 - x)$, $\sec(x) = \csc(\pi/2 - x)$.

AP Precalc Tip: Cosine is even, sine and tangent are odd. Cofunction identities convert between a function and its cofunction of a complementary angle.

Example: $\sin(\pi/2 - x) = \cos(x)$. This is why sin and cos graphs are shifted versions of each other.

122 Simplifying Trig Expressions

Explanation: Use identities (Pythagorean, double-angle, reciprocal) to rewrite a complicated expression in simpler terms --- the goal is often a single function.

AP Precalc Tip: Combine like functions, factor, and substitute identities to reduce. Look for opportunities to turn tan or sec into sin/cos first.

Example: Simplify $(\sin^2 x + \cos^2 x) / \cos x = 1 / \cos x = \sec x$.

123 Polar Coordinates

Explanation: A point can be given as (r, θ) in polar form: r = distance from origin, θ = angle from positive x-axis. Conversions: $x = r \cos(\theta)$, $y = r \sin(\theta)$.

AP Precalc Tip: Convert polar to rectangular with $x = r \cos \theta$, $y = r \sin \theta$. A negative r means the point is reflected across the origin.

Example: The polar point $(2, \pi/2)$ converts to rectangular $(0, 2)$.

124 Converting Polar to Rectangular

Explanation: Given (r, θ) : $x = r \cos(\theta)$, $y = r \sin(\theta)$. This is a direct application of the definitions of sine and cosine.

AP Precalc Tip: Plug r and θ into $x = r \cos$ and $y = r \sin$. Keep θ in the original unit (radians usually) and evaluate.

Example: $(r, \theta) = (3, \pi)$ gives $x = 3 \cos(\pi) = -3$, $y = 3 \sin(\pi) = 0$, so $(-3, 0)$.

125 Converting Rectangular to Polar

Explanation: Given (x, y) : $r = \sqrt{x^2 + y^2}$ and $\theta = \arctan(y/x)$, adjusted for the correct quadrant. Note r can be positive (sqrt) by convention.

AP Precalc Tip: Find r first as the distance $\sqrt{x^2 + y^2}$. For θ , use $\arctan(y/x)$ but add π (180) if $x < 0$ to land in the right quadrant.

Example: Point $(-1, 1)$: $r = \sqrt{2}$, $\theta = 3\pi/4$ (because it is in quadrant II).

126 Polar Equations: Circles

Explanation: $r = a$ is a circle of radius a centered at the origin. $r = a \cos(\theta)$ is a circle of diameter a centered on the x-axis; $r = a \sin(\theta)$ centered on the y-axis.

AP Precalc Tip: A constant r gives a circle about the pole. $r = a \cos/\sin$ give circles tangent to the origin on an axis.

Example: $r = 3$ is a circle radius 3 centered at the origin. $r = 2 \cos \theta$ is a circle of diameter 2 on the x-axis.

127 Polar Equations: Lines & Rays

Explanation: $\theta = \text{constant}$ is a ray (half-line) from the origin at that angle. A line through the origin is two opposite rays; lines not through the origin need other forms.

AP Precalc Tip: A fixed angle with r free gives a ray. Combine with other constraints for full lines. Recognize straight-line polar equations.

Example: $\theta = \pi/4$ is the ray at 45 degrees. Combined with $r \geq 0$ it is one ray; with all real r it is the full line.

128 Limacon & Rose Curves (overview)

Explanation: Polar graphs like $r = a + b \cos(\theta)$ (limacons) and $r = a \cos(k\theta)$ (rose curves with k petals) arise from parametric polar expressions.

AP Precalc Tip: These are often analyzed with a graphing calculator or by testing points. Know the family names and how the parameters shape them.

Example: $r = 3 + 3 \cos(\theta)$ forms a cardioid (a limacon with a cusp). $r = 2 \cos(2\theta)$ is a four-petal rose.

129 Graphing in Polar Coordinates

Explanation: To sketch $r = f(\theta)$, compute r for key angles ($0, \pi/2, \pi, 3\pi/2$) and plot points at distance r along angle θ . Join smoothly.

AP Precalc Tip: Systematically evaluate the polar function at quadrantal angles to get anchor points, then fill in the shape.

Example: $r = 2 \sin(\theta)$ at $\theta = 0$ gives $r = 0$ (origin), at $\pi/2$ gives $r = 2$ (top of circle) --- a circle of diameter 2.

130 Parametric & Polar Connection

Explanation: Polar coordinates can be seen as parametric: $x(t) = r(t)\cos(t)$, $y(t) = r(t)\sin(t)$ -type relations. Parametric thinking extends to vectors and motion.

AP Precalc Tip: Recognize that polar and parametric both describe points with a parameter (the angle or time). This unifies the final unit.

Example: On the unit circle, $x = \cos(t)$, $y = \sin(t)$ is both parametric motion and the polar point $(1, t)$.

131 Periodic Function Definition

Explanation: A function f is periodic with period p if $f(x + p) = f(x)$ for all x . The fundamental period is the smallest positive p .

AP Precalc Tip: To find the period from a graph, measure the distance for one complete cycle. For $y = a \sin(bx)$, period = $2\pi/|b|$.

Example: $\sin(x)$ has period 2π ; $\tan(x)$ has period π . The graph repeats every period.

132 Amplitude in Context

Explanation: Amplitude is the maximum displacement from the midline, equal to half the peak-to-trough distance.

AP Precalc Tip: Amplitude answers 'how far from average does the value swing?'. Extract it as $(\max - \min)/2$ from a graph or data.

Example: Values oscillating between 60 and 100 have amplitude 20 and midline 80.

133 Determining b from Period

Explanation: Given the period T of a sine/cosine function, $b = 2\pi/T$. This converts a period into the coefficient in $y = a \sin(bx)$.

AP Precalc Tip: If you can measure the period from the graph or context, divide 2π by it to get b . Shorter period = larger b .

Example: A cycle every π units: $b = 2\pi/\pi = 2$, so $y = \sin(2x)$.

134 Solving $\sin(x) = k$ over an Interval

Explanation: Use the unit circle to find the principal solution, then use symmetry (sin symmetric about $\pi/2$, cos about $0/2\pi$) to find all solutions in the interval.

AP Precalc Tip: Find one solution, then reflect for the other within one period. Add or walk to cover the requested interval.

Example: $\cos(x) = -1/2$ on $[0, 2\pi]$: $x = 2\pi/3$ and $x = 4\pi/3$.

135 Secant & Cosecant Graphs

Explanation: $\sec(x) = 1/\cos(x)$ and $\csc(x) = 1/\sin(x)$ have vertical asymptotes where their denominator function is zero, and their graphs are 'U' shapes between asymptotes.

AP Precalc Tip: sec has asymptotes where $\cos = 0$ ($\pi/2 + k\pi$); csc where $\sin = 0$ ($k\pi$). They are never between -1 and 1.

Example: $\sec(x)$ has asymptotes at $x = \pi/2$ and $3\pi/2$; between them the graph opens up or down.

136 Trig Formulas in Application

Explanation: Use $1/2 r^2 \theta$ for a sector and $s = r \theta$ for arc length in applied contexts like a clock hand or Ferris wheel radius.

AP Precalc Tip: These formulas are the practical payoff of radian measure. Match the quantity to the formula: arc-length uses $r \theta$, area uses $(1/2)r^2 \theta$.

Example: A clock's minute hand of length 4 sweeps an arc of length $4\pi/3$ in 20 minutes.

137 Polar to Rectangular and Back

Explanation: Rectangular (x, y) and polar (r, θ) convert with $x = r \cos \theta$, $y = r \sin \theta$, $r = \sqrt{x^2 + y^2}$, $\theta = \arctan(y/x)$ adjusted by quadrant.

AP Precalc Tip: Know both directions. For the angle, correct by quadrant: add π when $x < 0$. Keep r positive by using sqrt.

Example: Rectangular $(1,1)$ gives polar $(\sqrt{2}, \pi/4)$; polar $(2, \pi/3)$ gives rectangular $(1, \sqrt{3})$.

138 Graphing Basic Polar Curves

Explanation: Common polar curves: $r = a$ (circle radius a), $\theta = c$ (ray), $r = a \cos \theta$ / $r = a \sin \theta$ (diameter- a circles), $r = a + b \cos \theta$ (limacons).

AP Precalc Tip: Identify the family from the equation form. A cosine inside gives symmetry about the x -axis; sine about the y -axis.

Example: $r = 3$ is a circle radius 3; $r = 3 \cos \theta$ is a circle of diameter 3 centered at $(1.5, 0)$.

139 Modeling with Periodic Functions

Explanation: Real periodic data (tides, temperature, daylight, sound) are fit with $y = a \sin(b(x-h)) + d$.

AP Precalc Tip: Set d = midline, a = amplitude, $b = 2\pi/\text{period}$, h = phase alignment. Verify with a known data point.

Example: Tide levels oscillating between 2 and 10 m every 12 h: midline 6, amplitude 4, $b = \pi/6$.

Unit 4: Functions Involving Parameters, Vectors, and Matrices

15-30% of the exam. Parametric equations, vectors, and matrices.

140 Parametric Equations

Explanation: Parametric equations express x and y in terms of a third parameter t : $x = f(t)$, $y = g(t)$. They describe motion and curves that may not be functions $y(x)$.

AP Precalc Tip: Eliminate t by solving one equation for t and substituting, or use a trig identity for trig parametrics. A curve may loop or reverse --- read over the parameter interval.

Example: $x = t$, $y = t^2$ gives the parabola $y = x^2$. $x = \cos(t)$, $y = \sin(t)$ traces the unit circle.

141 Parameterizing Lines & Motions

Explanation: A line through (x_0, y_0) with direction (a, b) : $x = x_0 + a t$, $y = y_0 + b t$. Increasing t moves along the direction vector.

AP Precalc Tip: The direction vector (a, b) is the slope's vector form; the speed is $\sqrt{a^2 + b^2}$ per unit t . Position at t is the point on the line.

Example: Through $(1, 2)$ with direction $(3, 4)$: $x = 1 + 3t$, $y = 2 + 4t$. At $t = 1$, position $(4, 6)$.

142 Eliminating the Parameter

Explanation: To convert parametric to rectangular, solve the easier equation for t and substitute, or square and add for sine/cosine forms. State the domain of the rectangular curve.

AP Precalc Tip: After eliminating t , the resulting curve may have a restricted domain/range --- note it. This is a common place to lose points.

Example: $x = 2t$, $y = 6t^2$: $t = x/2$, so $y = 6(x/2)^2 = (3/2)x^2$, x any real.

143 Parametric Circles & Ellipses

Explanation: A circle: $x = r \cos(t)$, $y = r \sin(t)$. An ellipse: $x = a \cos(t)$, $y = b \sin(t)$. The parameter t is the angle of rotation.

AP Precalc Tip: For an ellipse, a and b are the radii along x and y . Use $\cos^2 + \sin^2 = 1$ to eliminate t and confirm $(x/a)^2 + (y/b)^2 = 1$.

Example: $x = 3 \cos(t)$, $y = 2 \sin(t)$ is the ellipse $x^2/9 + y^2/4 = 1$.

144 Speed Along a Parametric Path

Explanation: Speed = $\sqrt{(dx/dt)^2 + (dy/dt)^2}$. It is the magnitude of the velocity vector and equals the instantaneous rate of distance traveled.

AP Precalc Tip: Compute derivatives dx/dt and dy/dt , square them, add, and take the square root. Speed is always nonnegative (unlike velocity along an axis).

Example: If $dx/dt = 3$ and $dy/dt = 4$, speed = $\sqrt{9 + 16} = 5$.

145 Vectors: Basics

Explanation: A vector has magnitude and direction, written $\mathbf{v} = (a, b)$ or $a\mathbf{i} + b\mathbf{j}$. Magnitude $|\mathbf{v}| = \sqrt{a^2 + b^2}$. It is a displacement or directed quantity.

AP Precalc Tip: Distinguish a vector (magnitude + direction) from a scalar (just magnitude). Write in component or $\mathbf{i}$ - $\mathbf{j}$ form consistently.

Example: Vector $\mathbf{v} = (3, 4)$ has magnitude $\sqrt{9+16} = 5$. It has a direction of $\arctan(4/3)$ above the x -axis.

146 Vector Addition & Subtraction

Explanation: Add vectors component-wise: $(a,b) + (c,d) = (a+c, b+d)$. Subtract similarly. Graphically, use tip-to-tail for addition.

AP Precalc Tip: Add/subtract only like components. Geometrically, $\mathbf{v} - \mathbf{w}$ is the vector from the tip of $\mathbf{w}$ to the tip of $\mathbf{v}$.

Example: $(2,3) + (1,-1) = (3,2)$. $(2,3) - (1,-1) = (1,4)$.

147 Scalar Multiplication

Explanation: Multiplying a vector by a scalar k changes its magnitude by $|k|$ and flips direction if $k < 0$: $k(a,b) = (ka, kb)$.

AP Precalc Tip: Scalar multiplication scales each component. A negative scalar reverses direction. Two vectors are parallel if one is a scalar multiple of the other.

Example: $3(2,1) = (6,3)$; $-2(2,1) = (-4,-2)$ (opposite direction, double magnitude).

148 Unit Vectors

Explanation: A unit vector has magnitude 1. To find it, divide the vector by its magnitude: $u = v/|v|$. In i-j form, $i = (1,0)$ and $j = (0,1)$ are the standard unit vectors.

AP Precalc Tip: The unit vector is direction without scale. Any nonzero $v = |v| \cdot u$. Use it to preserve direction while normalizing length.

Example: For $v = (3,4)$, $|v| = 5$, so the unit vector is $(3/5, 4/5)$.

149 Vector Magnitude

Explanation: $|v| = \sqrt{a^2 + b^2}$ for $v = (a,b)$. It is the length of the vector and the distance between its endpoints when drawn from the origin.

AP Precalc Tip: Magnitude is always nonnegative and equals the distance formula in disguise. It is the hypotenuse of the component right triangle.

Example: The vector $(6,8)$ has magnitude 10 (a 6-8-10 triangle).

150 Dot Product

Explanation: $u \cdot v = a_1a_2 + b_1b_2$ (scalar). Equivalently $u \cdot v = |u||v| \cos(\theta)$ where θ is the angle between them.

AP Precalc Tip: The dot product returns a scalar, not a vector. It is zero when the vectors are perpendicular ($\cos 90^\circ = 0$).

Example: $(2,3) \cdot (4,-1) = 8 - 3 = 5$. Vectors $(1,2)$ and $(2,-1)$ are perpendicular because $2 - 2 = 0$.

151 Angle Between Vectors

Explanation: $\cos(\theta) = (u \cdot v) / (|u||v|)$. Solve for θ to find the angle between two vectors. Perpendicular iff $\cos \theta = 0$.

AP Precalc Tip: Use the two dot-product formulas together. If the dot product is 0 the vectors are orthogonal; positive means acute; negative obtuse.

Example: If $u \cdot v = 0$, then $\theta = 90^\circ$ (perpendicular).

152 Projection & Orthogonal Components

Explanation: The projection of v onto u is $\text{proj}_u(v) = (v \cdot u / |u|^2) u$. It measures how much of v points along u .

AP Precalc Tip: The projection is a vector parallel to u . The component of v perpendicular to u is $v - \text{proj}_u(v)$. These split v into orthogonal pieces.

Example: Projecting $(3,4)$ onto $(1,0)$ gives $(3,0)$ --- the horizontal component.

153 Vectors in Physics: Force & Motion

Explanation: Forces, velocities, and displacements are vectors. Resultant = vector sum. Break into components to analyze 2D motion.

AP Precalc Tip: Resolve each vector into x and y components, sum the components, then recombine: net force = $(\sum F_x, \sum F_y)$.

Example: $F_1 = (3,0)$, $F_2 = (0,4)$: resultant force = $(3,4)$ with magnitude 5.

154 Vector Applications: Navigation

Explanation: Velocity of an object + velocity of the medium (wind/current) combine by vector addition to give the resultant ground velocity.

AP Precalc Tip: Add the object's velocity vector to the medium's velocity vector. The ground speed is the magnitude of the resultant.

Example: A plane flying east $(300,0)$ with wind $(50,20)$: ground velocity $(350,20)$, ground speed ≈ 350.6 .

155 Matrices: Definition & Notation

Explanation: A matrix is a rectangular array of numbers arranged in rows and columns, denoted A with entries a_{ij} . Size = rows $\times$ columns.

AP Precalc Tip: Identify the size (rows $\times$ columns) and read entries by row/column. Matrices organize systems and transformations.

Example: A 2×3 matrix has 2 rows and 3 columns; entry a_{23} is row 2, column 3.

156 Matrix Addition & Scalar Multiplication

Explanation: Add matrices of the SAME size by adding corresponding entries. Multiply a matrix by a scalar by multiplying every entry.

AP Precalc Tip: Matrix addition requires matching dimensions. Scalar multiplication multiplies each entry by the scalar (entry-wise).

Example: $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$. $2 * \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$.

157 Matrix Multiplication

Explanation: Multiply A (m x n) by B (n x p) to get C (m x p): entry c_{ij} = dot product of row i of A with column j of B. Inner dimensions must match.

AP Precalc Tip: Matrix multiplication is NOT commutative ($AB \neq BA$ generally). The number of columns of A must equal the rows of B.

Example: $[1\ 2][a\ c] = [1a + 2c]$ (a 1x1). The inner dimension n must match.

158 Identity & Zero Matrices

Explanation: The identity matrix I has 1s on the diagonal and 0s elsewhere; $A^*I = I^*A = A$. The zero matrix has all zeros; $A + 0 = A$.

AP Precalc Tip: Multiply by the identity to leave a matrix unchanged (like multiplying by 1). The identity is square (same rows and columns).

Example: $I_2 = [1\ 0; 0\ 1]$. $[1\ 2; 3\ 4] * I_2 = [1\ 2; 3\ 4]$.

159 Determinants

Explanation: For a 2x2 matrix $[a\ b; c\ d]$, the determinant is $ad - bc$. It tells whether the matrix is invertible (nonzero determinant) and relates to area scaling.

AP Precalc Tip: A nonzero determinant means the matrix is invertible. A zero determinant means it is singular (not invertible) --- the transformation collapses area.

Example: $\det[3\ 4; 2\ 1] = 3*1 - 4*2 = 3 - 8 = -5$ (nonzero, invertible).

160 Matrix Inverses (2x2)

Explanation: The inverse of $[a\ b; c\ d]$ is $(1/(ad-bc)) * [d\ -b; -c\ a]$, provided $ad - bc \neq 0$. $A * A^{-1} = I$.

AP Precalc Tip: Divide by the determinant (must be nonzero). Swap a,d; negate b,c. Use it to solve matrix equations by multiplying by the inverse.

Example: For $[2\ 0; 0\ 3]$, $\det = 6$, inverse = $(1/6)[3\ 0; 0\ 2] = [1/2\ 0; 0\ 1/3]$.

161 Solving Systems with Matrices

Explanation: A system $AX = B$ is solved by $X = A^{-1}B$ when A is invertible. This is the matrix method for solving linear systems.

AP Precalc Tip: Set up coefficients in A and constants in B, compute A^{-1} , multiply. Requires A invertible ($\det \neq 0$, independent equations).

Example: Solve $2x = 4$: $A = [2]$, $X = [x]$, $B = [4]$; $x = A^{-1}B = (1/2)(4) = 2$.

162 Matrix Transformations

Explanation: A 2x2 matrix acts as a linear transformation on vectors: multiplying by a matrix rotates, scales, shears, or reflects vectors.

AP Precalc Tip: Think of a matrix as a rule that moves every point/vector; the determinant gives area scaling; special matrices rotate or reflect.

Example: The matrix $[0\ -1; 1\ 0]$ rotates vectors 90 degrees counterclockwise.

163 Rotation Matrix

Explanation: Rotating by angle theta: $R = [\cos \theta\ -\sin \theta; \sin \theta\ \cos \theta]$. Multiply a vector by R to rotate it by theta.

AP Precalc Tip: This matrix rotates points CCW by theta. It is used to animate rotations and understand transformation geometry.

Example: $R(90) = [0\ -1; 1\ 0]$; (1,0) rotates to (0,1).

164 Reflection Matrices

Explanation: Reflect across the x-axis: $[1\ 0; 0\ -1]$. Across the y-axis: $[-1\ 0; 0\ 1]$. Across $y=x$: $[0\ 1; 1\ 0]$.

AP Precalc Tip: These standard matrices flip points. Combine with rotation matrices to describe net transformations.

Example: Reflecting (2,3) across the x-axis: $[1\ 0; 0\ -1](2,3) = (2,-3)$.

165 Parametric Motion with Vectors

Explanation: Position $P(t) = (x(t), y(t))$ can be written as a position vector. Velocity is the derivative, acceleration the second derivative.

AP Precalc Tip: Differentiate each component independently for velocity/acceleration vectors. Speed is the magnitude of velocity.

Example: If $P(t) = (t^2, 3t)$, velocity = $(2t, 3)$, acceleration = $(2, 0)$.

166 Scalar vs Vector Quantities

Explanation: Scalars (magnitude only): temperature, speed, mass. Vectors (magnitude + direction): velocity, force, displacement, acceleration.

AP Precalc Tip: *Speed is a scalar (magnitude of velocity); velocity is a vector. This distinction is tested in context questions.*

Example: A car moves at 60 mph (speed, scalar) going north (velocity, vector).

167 Composition of Transformations

Explanation: Applying matrix M1 then M2 corresponds to multiplying by $M_2 \cdot M_1$ (rightmost applied first: $P' = M_2(M_1 P)$).

AP Precalc Tip: *Transformations compose right-to-left: the matrix closest to the vector is applied first. Order matters because matrices don't commute.*

Example: To rotate then reflect, multiply reflect * rotate * P --- apply rotate first.

168 Modeling with Parametric/Vectors/Matrices

Explanation: Real contexts (projectile motion, forces, computer graphics transforms, population matrices) are modeled by these tools.

AP Precalc Tip: *Recognize which tool fits: motion -> parametrics/vectors; forces -> vector sums; graphics/linear systems -> matrices.*

Example: A projectile path is modeled parametrically: $x = v_0 \cos(\theta) t$, $y = v_0 \sin(\theta) t - (1/2) g t^2$.

169 Graphing Parametric Curves

Explanation: To graph, compute (x, y) for a sequence of t values and connect the points in order of increasing t. Direction arrows show increasing t.

AP Precalc Tip: *Sample t over the given interval; the curve is traced in the direction of increasing t. Identify start/end points and direction.*

Example: For $x = t$, $y = t^2$ over t in [0,2], plot (0,0), (1,1), (2,4) --- an upward-right arc.

170 Parametric vs Cartesian (converting & limits)

Explanation: Convert a parametric path to Cartesian when asked for a rectangular equation, but remember the parameter may restrict the portion traced.

AP Precalc Tip: *Eliminate t for a Cartesian form, then determine the domain/range from the parameter interval. Include only the traced portion.*

Example: $x = \cos(t)$, $y = \sin(t)$ on t in [0, $\pi/2$] traces only the first-quadrant quarter of the unit circle.

171 Symmetric & Periodic Motion

Explanation: Periodic parametrics (trig-based) trace closed curves like circles and ellipses; understanding the parameter range reveals where and how many times they repeat.

AP Precalc Tip: *If both x and y are trig functions of t with a common period, the path is closed; the parameter interval tells how many full loops.*

Example: $x = \cos(2t)$, $y = \sin(2t)$ over t in [0, 2π] traces the circle twice.

172 Scalar Projection

Explanation: The scalar projection of v onto u is $(v \cdot u)/|u|$, a scalar giving the length of v's shadow along u.

AP Precalc Tip: *Distinguish the scalar projection (a number) from the vector projection (a vector parallel to u). Use |u|, not |v|, in the denominator.*

Example: For $v = (3,4)$ and $u = (1,0)$, the scalar projection of v onto u is 3.

173 Vector Addition in Context (Resultant)

Explanation: Combine displacement, force, or velocity vectors by adding components to get a resultant vector.

AP Precalc Tip: *Break each vector into x and y components, sum each, and recombine. The resultant's magnitude is $\sqrt{(\sum x)^2 + (\sum y)^2}$.*

Example: Walking 3 blocks east then 4 north gives resultant displacement (3,4), magnitude 5.

174 Matrix Product Dimensions

Explanation: The product of an $m \times n$ matrix and an $n \times p$ matrix is $m \times p$. The inner dimensions must match.

AP Precalc Tip: *Check dimensions first: columns of the first must equal rows of the second. The result's size is outer dimensions.*

Example: A 2×3 times a 3×4 gives a 2×4 matrix. A 3×3 times a 4×4 is undefined.

175 Matrix Multiplication Non-Commutativity

Explanation: Matrix multiplication is generally not commutative: AB is usually not equal to BA . Order reflects the order of transformations.

AP Precalc Tip: Do not reorder matrix products. In transformation contexts, the rightmost matrix is applied first.

Example: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ times $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ differs from the reverse order.

176 Finding a Matrix Inverse (2x2)

Explanation: For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $A^{-1} = \frac{1}{ad-bc}$ times $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$, if $ad-bc \neq 0$. Verify by checking $A \cdot A^{-1} = I$.

AP Precalc Tip: Swap a and d , negate b and c , and divide by the determinant. If the determinant is zero, the matrix has no inverse.

Example: For $\begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}$, $\det = 24 - 14 = 10$, so $A^{-1} = (1/10)\begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix}$.

177 Interpreting Matrix Transformations

Explanation: A 2x2 matrix maps vectors via linear transformation: columns describe where the basis vectors $(1,0)$ and $(0,1)$ go.

AP Precalc Tip: Read the transformed basis from the columns. The determinant's absolute value is the area scaling factor.

Example: The matrix $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ stretches the x -direction by 2 (sends e_1 to $(2,0)$).

178 Parametric Equations of Motion

Explanation: Projectile and other motion are modeled parametrically: $x(t)$ and $y(t)$ give position over time; they separate horizontal and vertical motion.

AP Precalc Tip: Model x with constant velocity ($v_0 \cos \theta$) and y with acceleration (gravitational $-g$), then combine for the path.

Example: Projectile: $x = 10t$, $y = 10 \sqrt{3} t - 4.9 t^2$ models a launch at 60 degrees.

179 Speed & Velocity from a Vector

Explanation: Speed is the scalar magnitude of the velocity vector; velocity includes direction. Both derive from the derivative of position.

AP Precalc Tip: $\text{Speed} = \sqrt{(dx/dt)^2 + (dy/dt)^2}$. Give speed as a nonnegative number and velocity as a vector with direction.

Example: If a particle's velocity is $(3, -4)$, its speed is 5 and it moves down-right.

180 Linear Transformations & Determinants

Explanation: The determinant equals the factor by which a transformation scales area; a negative determinant also reflects the orientation.

AP Precalc Tip: A zero determinant collapses the plane onto a line (not invertible). A determinant of 1 preserves area (rotation).

Example: The rotation matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ has determinant 1, preserving area while rotating 90 degrees.

What's Next?

You have now reviewed every unit of the AP Precalculus course. The concepts that matter most are the ones you almost knew -- go back to the Table of Contents, find the units you marked, and reread just those tip lines. Two weeks out, do a rapid pass over the entire guide; the day before, review only the tip lines. On exam day, practice with your calculator on, watch your units (radians vs degrees), and always check your domain and range answers.

Pair it with the Study Mastery Cheat Sheet (\$19) -- how to actually retain all of it

Also available: AP Calculus AB, AP Calculus BC, and AP Statistics study guides

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